

8.4 – Matrices for General Linear Transformations

Recall this example from section 1.8

Find the standard matrix A for the linear transformation $T : R^2 \rightarrow R^3$ for which

$$T \left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } T \left(\begin{bmatrix} 3 \\ -5 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -7 \\ 1 \end{bmatrix} \text{ and use it to compute } T \left(\begin{bmatrix} -4 \\ 3 \end{bmatrix} \right).$$

And #3 from 4.7

Consider the bases $B = \{u_1, u_2, u_3\}$ and $B' = \{u'_1, u'_2, u'_3\}$ for R^3 , where

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } u'_1 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, u'_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, u'_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

a. Find the transition matrix B to B'.

b. Compute the coordinate vector $[w]_B$, where $w = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$ and use the transition

matrix in part (a) to compute $[w]_{B'}$.

Now we combine these two concepts.

#5 Let $T : R^2 \rightarrow R^3$ be defined by $T \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_2 \\ -x_1 \\ 0 \end{bmatrix}$. *formula*

a. Find the matrix $[T]_{B',B}$ relative to the bases $B = \{u_1, u_2\}$ and $B' = \{v_1, v_2, v_3\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, u_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

We find $[T]_{B',B}$ using an augmented matrix that has new basis vectors as columns

[new / images of old]

on the left and images of old basis vectors as columns on the right:

a) We can find $[T]_{B',B}$ as follows.

First, find $T(\vec{u}_1)$ and $T(\vec{u}_2)$.

$$T(\vec{u}_1) = T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1+2\cdot 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$$

$$T(\vec{u}_2) = T\left(\begin{bmatrix} -2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -2+2\cdot 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 7 & 6 \\ 1 & 2 & 0 & -1 & 2 \\ 1 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & 8/3 & 4/3 \end{array} \right]$$

$$[T]_{B',B} = \left[\begin{array}{c|c} [T(\vec{u}_1)]_{B'} & [T(\vec{u}_2)]_{B'} \end{array} \right] = \begin{bmatrix} 0 & 0 \\ -1/2 & 1 \\ 8/3 & 4/3 \end{bmatrix}$$

This is the standard matrix for the transformation T relative to the bases B and B' .

It acts on coordinate vectors relative to B and returns coordinate vectors relative to B' .

b. Verify that the formula $[T]_{B',B} [\mathbf{x}]_B = [T(\mathbf{x})]_{B'}$ holds for every vector in \mathbb{R}^2 .

LHS: • Find $[\vec{x}]_B$ for an arbitrary vector in \mathbb{R}^2 .

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & x_1 \\ 3 & 4 & x_2 \end{array} \right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{2x_1 + x_2}{5} \\ 0 & 1 & \frac{-3x_1 + x_2}{10} \end{array} \right] \Rightarrow [\vec{x}]_B = \begin{bmatrix} \frac{2x_1 + x_2}{5} \\ \frac{-3x_1 + x_2}{10} \end{bmatrix}$$

• Multiply: $\begin{bmatrix} 0 & 0 \\ -1/2 & 1 \\ 8/3 & 4/3 \end{bmatrix} \begin{bmatrix} \frac{2x_1 + x_2}{5} \\ \frac{-3x_1 + x_2}{10} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2}x_1 \\ \frac{2}{3}x_1 + \frac{2}{3}x_2 \end{bmatrix}$

$$\text{RHS: } T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ -x_1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 + 2x_2 \\ 1 & 2 & 0 & -x_1 \\ 1 & 0 & 0 & 0 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2}x_1 \\ 0 & 0 & 1 & \frac{2}{3}x_1 + \frac{2}{3}x_2 \end{array} \right]$$

Match

For a specific example, suppose we want use $[T]_{B',B}$ to find $T\left(\begin{bmatrix} 11 \\ 3 \end{bmatrix}\right)$, where

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ -x_1 \\ 0 \end{bmatrix}. \quad T\left(\begin{bmatrix} 11 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 11 + 2 \cdot 3 \\ -11 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 \\ -11 \\ 0 \end{bmatrix}$$

Step 1: Express $\vec{x} = (11, 3)$ in terms of B .

$$\left[\begin{array}{cc|c} 1 & -2 & 11 \\ 3 & 4 & 3 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \end{array} \right]$$

$$\text{so } \begin{bmatrix} 11 \\ 3 \end{bmatrix}_B = \begin{bmatrix} 5 \\ -3 \end{bmatrix} = [\vec{x}]_B$$

Step 2: Multiply

$$[T]_{B',B} [\vec{x}]_B = \begin{bmatrix} 0 & 0 \\ -1/2 & 1 \\ 8/3 & 4/3 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -11/2 \\ 28/3 \end{bmatrix} = [T(\vec{x})]_{B'}$$

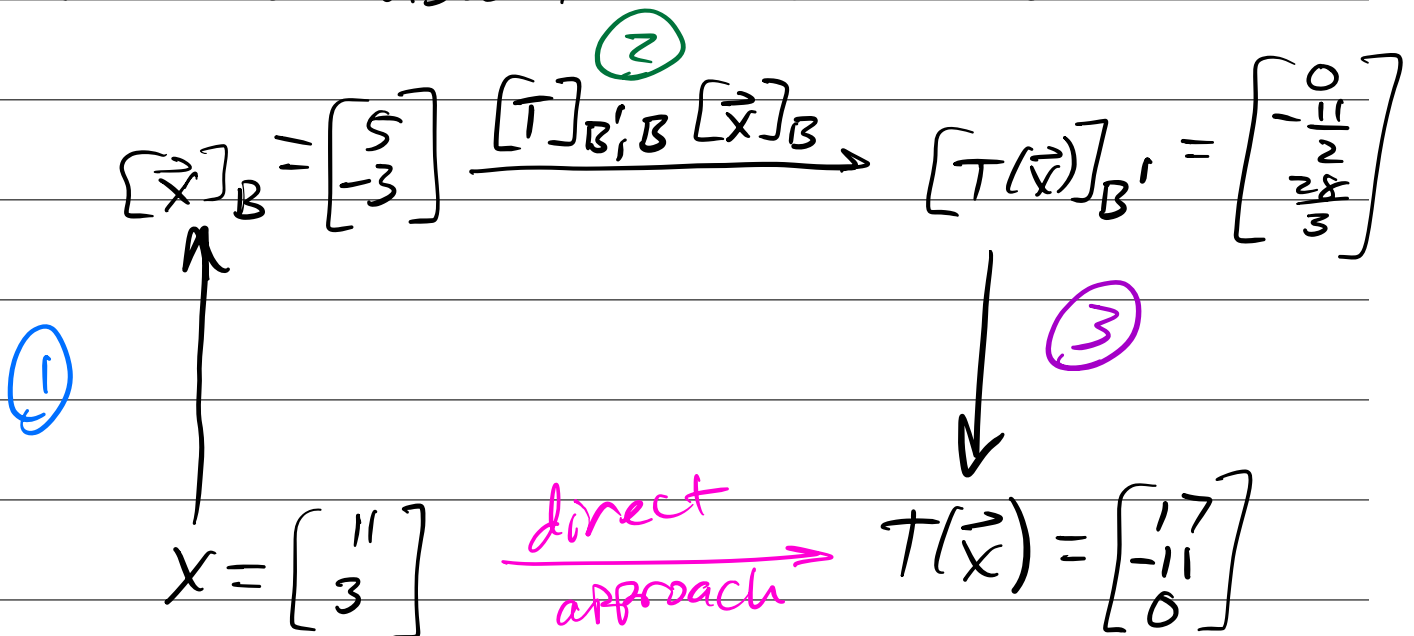
Step 3: Use the new coordinate vector with B'

$$T(\vec{x}) = 0\vec{v}_1 - \frac{11}{2}\vec{v}_2 + \frac{28}{3}\vec{v}_3$$

$$= -\frac{11}{2} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + \frac{28}{3} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 17 \\ -11 \\ 0 \end{bmatrix}$$

Recap: $(11, 3)$ *row reduce* $(5, -3)$ *multiply* $(0, -\frac{11}{2}, \frac{28}{3})$ *lin comb* $(17, -11, 0)$
 \vec{x} ① $[\vec{x}]_B$ ② $[T(\vec{x})]_{B'}$ ③

This can be visualized as follows



Let V be an n -dimensional vector space with basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ and let W be an m -dimensional vector space with basis B' . Suppose that $T : V \rightarrow W$ is a linear transformation, and that for each vector \mathbf{x} in V , the coordinate vectors for \mathbf{x} and $T(\mathbf{x})$ are $[\mathbf{x}]_B$ and $[T(\mathbf{x})]_{B'}$, respectively. Then the **matrix for T relative to the bases B and B'** is written [by this author] as

$$[T]_{B', B} = [[T(\mathbf{u}_1)]_{B'} | [T(\mathbf{u}_2)]_{B'} | \dots | [T(\mathbf{u}_n)]_{B'}].$$

This matrix has the property that $[T]_{B', B} [\mathbf{x}]_B = [T(\mathbf{x})]_{B'}$.

$$B: \vec{p}_1 = p_1(x) = 1, \quad \vec{p}_2 = p_2(x) = x, \quad \vec{p}_3 = p_3(x) = x^2$$

#8 Let $T: P_2 \rightarrow P_3$ be the linear transformation defined by $T(p(x)) = xp(x-3)$, that is, $T(c_0 + c_1x + c_2x^2) = x(c_0 + c_1(x-3) + c_2(x-3)^2)$

- Find $[T]_{B',B}$ relative to the bases $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3\}$.
- Use the three-step procedure illustrated in Example 2 to compute $T(1+x-x^2)$.
- Check the result obtained in part (b) by computing $T(1+x-x^2)$ directly.

a) $\left[\begin{array}{c|ccc} \text{new} & & & \\ \text{basis} & & & \\ \text{vectors} & & & \\ \hline & \text{images} & & \\ & \text{of old} & & \\ & \text{basis vectors} & & \end{array} \right]$

$$T(1) = x, \quad T(x) = x^2 - 3x, \quad T(x^2) = x(x-3)^2 = x^3 - 6x^2 + 9x$$

$= 0 + x + 0x^2 + 0x^3$
 $= 0 - 3x + x^2 + 0x^3$

$$\left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 & 9 \\ 0 & 0 & 1 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow [T]_{B',B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

b) $1+x-x^2 \rightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}_B$ (B is std basis)

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -11 \\ 7 \\ -1 \end{bmatrix} = [T(\vec{p})]_{B'}$$

std basis for P

$$\Rightarrow T(\vec{p}) = -11x + 7x^2 - x^3$$

c) $T(1+x-x^2) = x[1+(x-3)-(x-3)^2] = x(1+x-3-x^2+6x-9)$
 $= -11x + 7x^2 - x^3 \checkmark$

In the special case where $T: V \rightarrow V$ is a linear operator with

$B = B' = \{u_1, u_2, \dots, u_n\}$, the matrix for T is called the **matrix for T relative to the basis B** and is written [by this author] as $[T]_B = [[T(u_1)]_B | [T(u_2)]_B | \dots | [T(u_n)]_B]$.

old basis = new basis

#18 Let $D : P_2 \rightarrow P_2$ be the differentiation operator $D(p) = p'(x)$.

a. Find the matrix for D relative to the basis $B = \{p_1, p_2, p_3\}$ for P_2 in which $p_1 = 2$, $p_2 = 2 - 3x$, $p_3 = 2 - 3x + 8x^2$.

b. Use the matrix in part (a) to compute $D(6 - 6x + 24x^2)$.

a) $\left[\begin{array}{c|c} \text{new basis} & \text{images} \\ & \text{old basis} \end{array} \right] \quad D(\vec{p}_1) = \frac{d}{dx}(2) = 0,$

$D(\vec{p}_2) = \frac{d}{dx}(2 - 3x) = -3, \quad D(\vec{p}_3) = \frac{d}{dx}(2 - 3x + 8x^2) = -3 + 16x$

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 0 & -3 & -3 \\ 0 & -3 & -3 & 0 & 0 & 16 \\ 0 & 0 & 8 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -3/2 & 23/6 \\ 0 & 1 & 0 & 0 & 0 & -16/3 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right]$$

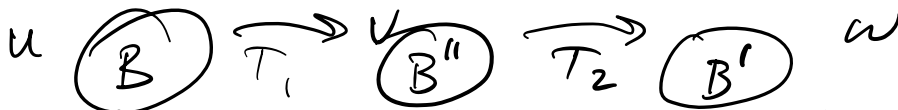
$$[D]_B = \begin{bmatrix} 0 & -3/2 & 23/6 \\ 0 & 0 & -16/3 \\ 0 & 0 & 0 \end{bmatrix}$$

b) $\left[\begin{array}{ccc|c} 2 & 2 & 2 & 6 \\ 0 & -3 & -3 & -6 \\ 0 & 0 & 8 & 24 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array} \right] \Rightarrow \begin{bmatrix} \vec{p} \\ \end{bmatrix}_B = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 0 & -3/2 & 23/6 \\ 0 & 0 & -16/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ -16 \\ 0 \end{bmatrix} = [D(\vec{p})]_B$$

$$D(\vec{p}) = 13(2) - 16(2 - 3x) + 0(2 - 3x + 8x^2) = 26 - 32 + 48x = -6 + 48x$$

Check: $\frac{d}{dx}(6 - 6x + 24x^2) = -6 + 48x \checkmark$



Theorem 8.4.1 If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are linear transformations, and if B, B'' , and B' are bases for U, V , and W , respectively, then $[T_2 \circ T_1]_{B', B} = [T_2]_{B', B''} [T_1]_{B'', B}$.

#12 Let $T_1 : P_1 \rightarrow P_2$ be the linear transformation defined by $T_1(p(x)) = xp(x)$ and let $T_2 : P_2 \rightarrow P_2$ be the linear operator defined by $T_2(p(x)) = p(2x+1)$. Let $B = \{1, x\}$ and $B' = \{1, x, x^2\}$ be the standard bases for P_1 and P_2 .

- Find $[T_2 \circ T_1]_{B', B}$, $[T_2]_{B', B'}$, and $[T_1]_{B', B}$.
- State a formula relating the matrices in part (a).
- Verify that the matrices in part (a) satisfy the formula you stated in part (b).

$$a) (T_2 \circ T_1)(\vec{p}) = T_2(T_1(\vec{p})) = T_2(xp(x)) = (2x+1)p(2x+1)$$

$$(T_2 \circ T_1)(1) = 2x+1, \quad (T_2 \circ T_1)(x) = (2x+1)^2$$

$$[T_2 \circ T_1]_{B', B} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad [T_2 \circ T_1]_{B', B} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}$$

$$[T_2 \circ T_1]_{B', B} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 0 & 4 \end{bmatrix}, \quad [T_2]_{B', B'} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$[T_1]_{B', B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad T: P_1 \rightarrow P_2$$

$$b) [T_2 \circ T_1]_{B', B} = [T_2]_{B', B'} [T_1]_{B', B}$$

$$c) [T_2]_{B', B'} [T_1]_{B', B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 0 & 4 \end{bmatrix}$$

$$= [T_2 \circ T_1]_{B', B}$$

Theorem 8.4.2

If $T : V \rightarrow V$ is a linear operator, and if B is a basis for V , then T is one-to-one if and only if $[T]_B$ is invertible. Moreover, when these conditions hold, $[T^{-1}]_B = [T]_B^{-1}$.